MATHEMATICS

NOTE ON A DUALITY RELATION OF KAASHOEK

BY

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The purpose of this note is to give a proof for a correct but incorrectly proven duality relation of KAASHOEK which appears in [2].

First a few preliminary definitions are needed. If T is a linear map with domain $D(T)$ in a vector space X and range $R(T)$ in a vector space Y then we will say that T acts from X to Y; when it is assumed that $D(T)=X$ we will say simply that T is from X to Y. If T acts from X to Y we define $\alpha(T)=\dim N(T)$, where $N(T)$ is the null space of T, and $\beta(T)=\dim Y/R(T)$, where here and below "dim" denotes the integer k if the vector space in question has finite dimension k and the extended real number $+\infty$ if it is infinite-dimensional. For each pair (T, S) where T and S act from X to Y we define a sequence $(D_n(T, S))_{n\geqslant 0}$ of subspaces of X as follows: $D_0(T, S)=X$, and for $n>0$, $D_n(T, S)=S^{-1}TD_{n-1}(T, S)$, where

$$TD_{n-1}(T, S)=\{Tx\colon x\in D(T)\cap D_{n-1}(T, S)\}$$

and

$$S^{-1}TD_{n-1}(T, S)=\{x\colon x\in D(S),\ Sx\in TD_{n-1}(T, S)\},$$

and we set $D(T, S)=\bigcap_{n\geqslant 0} D_n(T,S)$. If X and Y are normed vector spaces, T acts from X to Y, and $N(T)$ is a closed subspace of X, we define

$$\gamma(T)=\begin{cases}\inf\{\|Tx\|/d(x, N(T))\colon x\in D(T),\ x\notin N(T)\} \text{ if } T\neq 0,\\ +\infty \text{ if } T=0\end{cases}$$

where $d(x, N(T))$ denotes the distance from x to $N(T)$.

The following theorem is contained in Theorems 3 and 5 of the paper [1] by KATO:

Theorem 1. *Let X and Y be Banach spaces, T a closed linear map acting from X to Y with $0<\gamma(T)<\infty$, and S a linear map acting from X to Y with $D(S)\supset D(T)$ and $\|Sx\|\leqslant\sigma\|x\|+\tau\|Tx\|$ for some non-negative constants σ and τ and all $x\in D(T)$. Then the following statements hold:*

a) *If $N(T)\subset D(T, S)$ then there exists a constant $\varrho>0$ such that for $|\lambda|<\varrho$, $T-\lambda S$ is closed, $R(T-\lambda S)$ is closed, and $\alpha(T-\lambda S)$ and $\beta(T-\lambda S)$ are constant.*

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b) *If* $\alpha(T)$ *or* $\beta(T)$ *is finite then there exists a constant* $\varrho > 0$ *such that for* $0 < |\lambda| < \varrho$, $T - \lambda S$ *is closed*, $R(T - \lambda S)$ *is closed*, $\alpha(T - \lambda S) = \alpha(T) - k$, *and* $\beta(T - \lambda S) = \beta(T) - k$ *for some non-negative integer* k.

On examining Kato's proofs [1]) it can be seen that instead of b) the following stronger statement is actually established:

b′) *If* $N(T)/(N(T) \cap D(T, S))$ *has finite dimension* k *then there exists a constant* $\varrho > 0$ *such that for* $0 < |\lambda| < \varrho$, $T - \lambda S$ *is closed*, $R(T - \lambda S)$ *is closed*, $\alpha(T - \lambda S) = \alpha(T) - k$, *and* $\beta(T - \lambda S) = \beta(T) - k$.

In [2] KAASHOEK introduces the quantity $\dim N(T)/(N(T) \cap D(T, S))$ and proves Theorem 1 with statements a) and b′) using a method which yields for a) a better value for the constant ϱ than Kato's method; in fact, the value obtained is the best possible.

To prove b′) KAASHOEK uses the following duality relation:

$$\text{(1)} \qquad \dim N(T)/(N(T) \cap D(T, S)) = k \text{ if and only if } \dim N(T^*)/(N(T^*) \cap D(T^*, S^*)) = k,$$

where it is assumed that T and S are continuous linear maps from a Banach space X to a Banach space Y, $R(A)$ is closed, and $k < +\infty$. The proof of this relation is, however, based on a theorem (Theorem 3.3, p. 458 of [2]) which is not true. In this note we present a counterexample to this theorem and then prove a theorem which has (1) as a consequence.

Theorem 3.3 of [2] asserts the following:

Let C, T, and S be continuous linear maps from a Banach space X to a Banach space Y with $N(S) \subset N(C)$, $N(T) \subset D(T, S)$ and $\dim R(C) = l < +\infty$. Then $\dim N(T + C)/(N(T + C) \cap D(T + C, S)) \leqslant l$.

To see this is not true let $X = Y = l_2$ and define C, T, and S by $Ce_k = 0$, $k \neq 4$, $Ce_4 = -e_2 - e_4$; $Te_k = e_k$, $k \neq 3$, $Te_3 = 0$; and $Se_k = e_{k+1}$, $k \geqslant 1$; where e_k is the k-th unit vector $(\delta_{nk})_{n \geqslant 1}$. Let $D = [e_k]_{k \geqslant 3}$ (= closure of the subspace generated by the vectors e_k, $k \geqslant 3$) and $R = [e_k]_{k \geqslant 4}$. It is easily seen that C, T, and S are continuous with $N(S) = [0] \subset N(C)$, $\dim R(C) = 1$, and that $TD = R$, $S^{-1}R = D$, and $[e_3] = N(T) \subset D$, implying $N(T) \subset D(T, S)$. Hence the hypotheses are satisfied with $l = 1$. However, $(T + C)e_k = e_k$, $k \neq 3, 4$, $(T + C)e_3 = 0$, $(T + C)e_4 = -e_2$. Hence $(T + C)X = [e_k]_{k \neq 3,4}$,

$$S^{-1}(T + C)X = [e_k]_{k \neq 2,3},$$

and $N(T + C) = [e_2 + e_4, e_3]$. Thus $N(T + C) \cap S^{-1}(T + C)X = [0]$, implying $\dim N(T + C)/(N(T + C) \cap D(T + C, S)) = 2 > l = 1$.

The relation (1) is a consequence of the following theorem:

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[1]) Specifically it can be seen that Kato's Theorem 4, p. 307 of [1], holds if the hypothesis "$\alpha(A)$ or $\beta(A)$ is finite" is replaced by "$\dim N(A)/(N(A) \cap D(A, B)) = k < +\infty$" and that then the "$r$" of this theorem is equal to k; with this observation, statement b') follows by the proof of Theorem 5, p. 315 of [1].

Theorem 2. *Let T and S be continuous linear maps from a Banach space X to a Banach space Y with $R(T)$ closed. Then for all $n \geqslant 0$,*

$$\dim N(T)/(N(T) \cap D_n(T, S)) = \dim N(T^*)/(N(T^*) \cap D_n(T^*, S^*)).$$

Proof. Set $D_n = D_n(T, S)$, $D_n^* = D_n(T^*, S^*)$, and define a sequence $(N_n)_{n \geqslant 0}$ of subspaces of X as follows: $N_0 = \{0\}$, and for $n > 0$, $N_n = T^{-1}SN_{n-1}$. We first show

$$\dim N_n/(N_n \cap D_1) = \dim N_1/(N_1 \cap D_n), \quad n \geqslant 0. \tag{2}$$

Since

$$\dim N_n/(N_n \cap D_1) = \dim (N_n + D_1)/D_1 = \dim (N_n + D_1)/(N_{n-1} + D_1) + \dots + \dim (N_1 + D_1)/(N_0 + D_1),$$

and

$$\dim N_1/(N_1 \cap D_n) = \dim (N_1 \cap D_0)/(N_1 \cap D_1) + \dots + \dim (N_1 \cap D_{n-1})/(N_1 \cap D_n),$$

it is enough to show

$$\dim (N_{k+1} + D_1)/(N_k + D_1) = \dim (N_1 \cap D_k)/(N_1 \cap D_{k+1}), \quad 0 \leqslant k \leqslant n-1.$$

Let $(x_i^0)_{1 \leqslant i \leqslant n}$ be a sequence in $N_1 \cap D_k$ linearly independent modulo $N_1 \cap D_{k+1}$. Since it is in D_k there exist sequences $(x_i^j)_{1 \leqslant i \leqslant n} \subset D_{k-j}$, $1 \leqslant j \leqslant k$, such that

$$Sx_i^0 = Tx_i^1, Sx_i^1 = Tx_i^2, \dots, Sx_i^{k-1} = Tx_i^k. \tag{3}$$

Since it is also in N_1 we have from (3) that $(x_i^k)_{1 \leqslant i \leqslant n}$ is in N_{k+1}. Put $x^j = \sum_{i=1}^n \alpha_i x_i^j$, $0 \leqslant j \leqslant k$, and suppose $x^k \in N_k + D_1$. Then $x^k = n^k + d^1$ for some $n^k \in N_k$ and $d^1 \in D_1$, and hence $Sx^{k-1} = Tx^k = Tn^k + Td^1 = Sn^{k-1} + Td^1$ for some $n^{k-1} \in N_{k-1}$. Thus $S(x^{k-1} - n^{k-1}) = Td^1$, i.e., $x^{k-1} = n^{k-1} + d^2$ for some $d^2 \in D_2$. Proceeding in this way we get $x^1 = n^1 + d^k$ for some $n^1 \in N_1$ and $d^k \in D_k$ and hence $Sx^0 = Tx^1 = Td^k$, i.e., $x^0 \in D_{k+1}$. Since $(x_i^0)_{1 \leqslant i \leqslant n}$ is linearly independent modulo $N_1 \cap D_{k+1}$ it follows that $\alpha_1 = \dots = \alpha_n = 0$, proving $(x_i^k)_{1 \leqslant i \leqslant n}$ is linearly independent modulo $N_k + D_1$. Thus

$$\dim (N_{k+1} + D_1)/(N_k + D_1) \geqslant \dim (N_1 \cap D_k)/(N_1 \cap D_{k+1}).$$

To get the opposite inequality let $(x_i^0)_{1 \leqslant i \leqslant n}$ be a sequence in N_{k+1} linearly independent modulo $N_k + D_1$. Then there exist sequences

$$(x_i^j)_{1 \leqslant i \leqslant n} \subset N_{k-j+1}, \quad 1 \leqslant j \leqslant k$$

such that

$$Tx_i^0 = Sx_i^1, Tx_i^1 = Sx_i^2, \dots, Tx_i^{k-1} = Sx_i^k. \tag{4}$$

From (4) it follows that $(x_i^k)_{1 \leqslant i \leqslant n}$ is in D_k. Put $x^j = \sum_{i=1}^n \alpha_i x_i^j$, $0 \leqslant j \leqslant k$, and suppose $x^k \in D_{k+1}$. Then $Tx^{k-1} = Sx^k = Td^k$ for some $d^k \in D_k$, implying $x^{k-1} - d^k = n^1$ for some $n^1 \in N_1$. Similarly,

$$Tx^{k-2} = Sx^{k-1} = Sn^1 + Sd^k = Sn^1 + Td^{k-1}$$

for some $d^{k-1} \in D_{k-1}$ and hence $x^{k-2} \in D_{k-1} + N_2$. Repeating this procedure we finally get $x^0 \in N_k + D_1$, implying $\alpha_1 = \ldots = \alpha_n = 0$. Thus

$$\dim (N_{k+1} + D_1)/(N_k + D_1) \leqslant \dim (N_1 \cap D_k)/(N_1 \cap D_{k+1}).$$

We now show $D_n^* \subset (SN_n)^\perp$, $n \geqslant 0$. This is obvious for $n=0$. Suppose $f \in D^*_{n+1}$, $x \in N_{n+1}$, $g \in D_n^*$ is such that $S^*f = T^*g$, and $y \in N_n$ is such that $Tx = Sy$. Then $f(Sx) = (S^*f)x = (T^*g)x = g(Tx) = g(Sy)$. Thus $D_n^* \subset (SN_n)^\perp$ implies $D^*_{n+1} \subset (SN_{n+1})^\perp$, and the relation $D_n^* \subset (SN_n)^\perp$, $n \geqslant 0$, follows by induction.

With the relations $D_1 = S^{-1}R(T)$ and $R(T)^\perp = N(T^*)$, and the hypothesis that $R(T)$ is closed it is not difficult to show that

$$(5) \quad \dim N_n/(N_n \cap D_1) = \dim (R(T) + SN_n)/R(T) = \\ = \dim N(T^*)/(N(T^*) \cap (SN_n)^\perp).$$

If for some n (and hence for all larger n in view of $N_{n+1} \supset N_n$)

$$\dim N(T^*)/(N(T^*) \cap (SN_n)^\perp) = +\infty$$

let p be the least such n; otherwise set $p = +\infty$. Since, as we have already shown, $D_n^* \subset (SN_n)^\perp$ for all n,

$$(6) \qquad \dim N(T^*)/(N(T^*) \cap (SN_n)^\perp) = \dim N(T^*)/(N(T^*) \cap D_n^*)$$

holds for $n \geqslant p$. For $n < p$ we will prove (6) by showing $(SN_n)^\perp \subset D_n^*$, and thus complete the proof in view of (2) and (5). This is clear for $n=0$. Assume $(SN_n)^\perp \subset D_n^*$ and let $f \in (SN_{n+1})^\perp$. If there is a $g \in (SN_n)^\perp$ such that $T^*g = S^*f$, then from $(SN_n)^\perp \subset D_n^*$ and the definition of D^*_{n+1} it follows that $f \in D^*_{n+1}$, i.e., $(SN_{n+1})^\perp \subset D^*_{n+1}$. We now find such a g. Since $N(T) = N_1 \subset N_{n+1}$ we have $S^*f(N(T)) \subset S^*f(N_{n+1}) = \{0\}$. We can therefore define a linear functional g' on the range $R(T)$ by the relation $g'(Tx) = (S^*f)x$; g' is continuous because

$$|g'(Tx)| = \inf \{|g'(T(x+x'))| : x' \in N(T)\} \\ \leqslant \inf \{\|S^*f\| \cdot \|x+x'\| : x' \in N(T)\} \leqslant \|S^*f\| \cdot \gamma(T)^{-1} \cdot \|Tx\|.$$

Since $\dim (R(T) + SN_n)/R(T)$ is finite, there is a finite-dimensional subspace M of Y such that $SN_n = M \oplus (SN_n \cap R(T))$. Let P be the projection in the Banach space $M \oplus R(T)$ with null space M and range $R(T)$ and define $g'' = g' \circ P$. If $x \in N_n$ and $Sx \in R(T)$ we have $Sx = Tx'$ for some $x' \in N_{n+1}$, so that $g''(Sx) = g''(Tx') = g'(Tx') = f(Sx') = 0$. Since g'' annihilates

M we thus have $g''(SN_n) = \{0\}$. Finally, since g'' is continuous,

$$(\|g''\| \leqslant \|g'\| \cdot \|P\|),$$

there is a $g \in Y^*$ extending g'', and any such g obviously has the required properties. This completes the proof.

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